1. Given that $P(A) = a$ and $P(B) = b$, show that

$$P(A|B) \geq \frac{a+b-1}{b}.$$

$$\begin{aligned} P(A|B) = \frac{P(A \cap B)}{P(B)} &= \frac{P(A) + P(B) - P(A \cup B)}{P(B)} \\ &\geq \frac{P(A) + P(B) - 1}{P(B)} \end{aligned}$$

Since $1 \geq P(A \cup B)$ and subtracting bigger numbers makes things smaller.

2. An urn contains three white chips, six black chips, and four red chips. Four chips are drawn in order and without replacement: black, red, white, red. What is the probability of drawing this sequence of chips?

$P(B \cap R \cap W \cap R)$ can be written as $P(B)P(R|B)P(W|B \cap R)P(R|B \cap R \cap W)$, which is calculated as $\frac{6}{13}\frac{4}{12}\frac{3}{11}\frac{3}{10}$.

3. Consider the set of families having two children. Assume that the four possible birth sequences – (both are boys), (younger child is a boy, older child is a girl), (younger child is a girl, older child is a boy), and (both are girls) – are equally likely. What is the probability that both children are boys given (conditioned on) at least one is a boy? Hint: the answer is not 1/2.

Let $A = \{$ both boys $\}$ and $B = \{$ at least one is a boy $\}$. Then $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$, since the only place A and B intersect is with both two boy children. Now $P(A) = 1/4$ and $P(B) = 3/4$. Therefore, $P(A|B) = 1/3$.

4. A man has n keys on a key ring, one of which opens the door to his apartment. Having celebrated a bit too much one evening, he returns home only to find himself unable to distinguish one key from another. Resourceful, he works out a fiendishly clever plan: He will choose a key at random and try it. If it fails to open the door, he will discard it, and choose at random one of the remaining $n - 1$ keys, and so on. Clearly, the probability that he gains entrance with the first key he selects is $1/n$. Show that the probability the door opens with the third key he tries is also $1/n$.

We are interested in finding the quantity, $P(\text{not } 1 \cap \text{not } 2 \cap 3)$. With conditional probability we can write this as $P(\text{not } 1)P(\text{not } 2|\text{not } 1)P(3|\text{not } 1 \cap \text{not } 2)$. Since there are n keys, of which $n-1$ don't work, $P(\text{not } 1) = \frac{n-1}{n}$. After we discard this first not working key, there are $n - 1$ keys of which $n - 2$ don't work. Hence, $P(\text{not } 2|\text{not } 1) = \frac{n-2}{n-1}$. This third key is successful, after discarding the first two, $P(3|\text{not } 1 \cap \text{not } 2) = \frac{1}{n-2}$. Putting all this together we find, $P(\text{not } 1 \cap \text{not } 2 \cap 3) = \frac{n-1}{n}\frac{n-2}{n-1}\frac{1}{n-2} = \frac{1}{n}$.