1. Let $f(x) = \frac{6}{\pi^2 x^2}$ for $x = 1, 2, 3, \ldots,$. Show that $f(x)$ is a probability mass function. Show that $\mathbb{E}(X)$ does not exist.

   (a) $\sum_{x \in \mathbb{N}\setminus\{0\}} \frac{1}{x^2} = \frac{\pi^2}{6}$, which will make this a probability mass function.

   (b) $\mathbb{E}(X) = \frac{6}{\pi^2} \sum_{x \in \mathbb{N}\setminus\{0\}} \frac{x}{x^2} > \infty$, by the p-series convergence test.

2. Let X be a random variable with support $\{1, 2, 3, 5, 15, 25, 50\}$, for which each point has probability $1/7$. Argue that $c = 5$ is the value that minimizes $h(c) = \mathbb{E}(|X - c|)$. Compare with the value of b that minimizes $g(b) = \mathbb{E}[(X - b)^2]$.

   Pick a few values different than $c = 5$ and evaluate the expectations. Note any attempt to make the minimizing c smaller, adds to the difference from values above 5, and any attempt to make c bigger adds to the difference from values below 5. On the other hand, since the values to the right of 5 are so much larger from the "center", minimizing squared differences will pull the expected value up. The expected value is indeed larger than 5, $\mathbb{E}(X) = 14.43$.

3. Let X be a discrete uniform random variable on the bounds $[a, b]$ for $0 < a < b$, namely $X \sim U(a, b)$. Show $\mathbb{E}(X) = \frac{a+b}{2}$. The distribution function for X is

   $$f(x) = \frac{1}{b - a + 1}, \quad x = a, a + 1, ..., b - 1, b.$$

   Hint: What's the forumula for the sum of the first n positive integers?

   $\mathbb{E}(X) = \frac{1}{b-a+1} \sum_{x \in [a,b]} x$. To sum all the integers from 1 to n, we use the formula $\frac{n(n+1)}{2}$. Here, we want to sum all the integers from 1 to b and subtract from this the sum of all integers from 1 to $(a - 1)$.

   $$\begin{aligned} \mathbb{E}(X) &= \frac{1}{b - a + 1} \left(\frac{b(b + 1)}{2} - \frac{(a - 1)a}{2} \right) \\ &= \frac{(b - a)(b + a) + (b + a)}{b - a + 1} \\ &= \frac{b + a}{2} \end{aligned}$$

4. A fair coin is flipped until a head appears. You will be given $(1/2)^k$ dollars if the first head appears on the kth flip. What is the expected payout? Hint: first find the probability mass function, then determine the payout function and use it to calculate the general expected value.

   By independence of flips, the probability mass function if $f(k) = (1/2)^k$, which is the same as the payout.

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k \in \mathbb{N} \setminus \{0\}} (1/2)^{2k} \\
&= \sum_{k \in \mathbb{N} \setminus \{0\}} (1/4)^{k} \\
&= \sum_{k \in \mathbb{N} \setminus} (1/4)^{k} - (1/4)^{0} \\
&= \frac{1}{1 - 1/4} - 1 = \$0.33
\end{aligned}$$