1. Define the joint probability mass function of X, Y to be $f(x, y) = 1/5$ on the set of points $\{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$. Calculate the following

   (a) the marginal PMF of X, $f(x) = 1/5, \; x = -2, -1, 0, 1, 2$

   (b) the marginal PMF of Y,

   $$f(y) = \begin{cases} 1/5 & y = 0 \\ 2/5 & y = 1, 4 \end{cases}$$

   (c) $\mathbb{E}(XY) = \sum_{X,Y} x * y * f(x, y) = 0$

   (d) $\mathbb{E}(X) = \sum_X x * f(x) = 0$

   (e) $\mathbb{E}(Y) = \sum_Y y * f(y) = 2/5 + 8/5 = 2$

   (f) $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0$

2. If X and Y are independent random variables, what is the correlation between X and Y, namely ρ? Explain how you know this. We proved this in class.

3. If ρ, the correlation between X and Y, is equal to zero, are X and Y necessarily independent? Explain how you know this. No. All we need is one example to prove that two random variables can be correlated, but still have $\rho = 0$. Problem number 1 on this worksheet does the trick.

4. Let (X, Y) be a bivariate, continuous Uniform random variable defined on $0 < x < 1$ and $0 < y < 1$.

   (a) Draw the support of the joint random variable. Ask me me draw pictures in class, if you need.

   (b) Draw and shade the area defined by $x^2 + y^2 \leq 1$, contained within the support of (X, Y).

   (c) Define the random variable W as

   $$W = \begin{cases} 1 & \text{if } X^2 + Y^2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

   W is a univariate random variable with a common name. What is the name of the distribution of W? W is a Bernoulli random variable, $W \sim Bernoulli(p = \pi/4)$.

   (d) What is the probability that $P(W = 1) = \pi/4$?

   (e) What is the expected value of W, namely $\mathbb{E}(W) = \pi/4$.?

   (f) Provide a new random variable, P, as a function of W, who's expected value is π. $P = 4 * W$.

5. **Optional Challenge Question 1**. Let X and Y be defined such that $\mathbb{E}(X) = \mu_X$, $\mathbb{V}(X) = \sigma_X^2$ and $\mathbb{E}(Y) = \mu_Y$, $\mathbb{V}(Y) = \sigma_Y^2$. The values of (β_0, β_1) that minimize the function $K(\beta_0, \beta_1) = \mathbb{E}[(Y - (\beta_0 + \beta_1 * X))^2]$ are the values that define the least squares regression line – the line most often fit through a scatter plot of data. Find the values of (β_0, β_1) that minimize K. Show that $a = \mu_y - \mu_x b$ and $b = \text{Cov}(X, Y)/\sigma_X^2$.

6. **Optional Challenge Question 2**. Suppose X and Y are random variables such that $\mathbb{E}(X) = \mu_X$ and $\mathbb{E}(Y) = \mu_Y$. Using the short-cut formula for the variance prove that $\mathbb{V}(aX + bY) = a^2\mathbb{V}(X) + b^2\mathbb{V}(Y) + 2ab\text{Cov}(X, Y)$, for constants a, b.